Optimal Bidding in Power Markets: Tests for Bilinear Bilevel Programs

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1 Introduction

Until recently the electrical power industry was a vertically integrated structure controlled by government regulations. In order to promote a more competitive environment, numerous countries have moved to a deregulated electrical power industry. This deregulation has caused the separation of electrical power generation, transmission, distribution, and retail. The structure of this new power system is based on a market model to effect the sale of electricity using supply and demand to set the price.

The market model and mechanism by which prices are set and electricity dispatched can be described by the primal and dual pair of linear programs:

$$\begin{array}{llll} \underset{\mathbf{q}}{\text{minimize}} & \mathbf{p}^T\mathbf{q} & \underset{\boldsymbol{\pi},\mathbf{s}}{\text{maximize}} & \mathbf{b}^T\boldsymbol{\pi} \\ \text{subject to} & A\mathbf{q} = \mathbf{b} & \text{subject to} & A^T\boldsymbol{\pi} + \mathbf{s} = \mathbf{p} \\ & \mathbf{0} \le \mathbf{q} & & \mathbf{0} \le \mathbf{s} \end{array}$$

Here $\mathbf{p}$ is a vector of electric power bid prices, $\mathbf{q}$ is a vector of electric power bid quantities, and $\boldsymbol{\pi}$ is a vector of market prices. From the solution vectors $\mathbf{q}^\star$ and $\boldsymbol{\pi}^\star$ of these linear programs it is possible to calculate the market clearing quantity and market clearing price.

In this work we consider problems in which a firm or single entity in the market seeks to maximize its profit subject to the constraint that the market is in equilibrium, or that $\mathbf{q}, \boldsymbol{\pi}$ and $\mathbf{s}$ are optimal vectors for the above linear programs. Thus, we consider problems in the form

$$\begin{array}{l} \underset{\mathbf{p},\mathbf{b}}{\text{maximize}} \quad f(\mathbf{p}, \mathbf{q}^\star, \boldsymbol{\pi}^\star) \\ \text{subject to} \\ \qquad \begin{array}{llll} \underset{\mathbf{q}}{\text{minimize}} & \mathbf{p}^T\mathbf{q} & \underset{\boldsymbol{\pi},\mathbf{s}}{\text{maximize}} & \mathbf{b}^T\boldsymbol{\pi} \\ \text{subject to} & A\mathbf{q} = \mathbf{b} & \text{subject to} & A^T\boldsymbol{\pi} + \mathbf{s} = \mathbf{p} \\ & \mathbf{0} \le \mathbf{q} & & \mathbf{0} \le \mathbf{s} \end{array} \cdot \end{array} \tag{1.1}$$

Here, f is a scalar-valued bilinear function that gives the profit of a firm or single entity in the market. Note the bilevel nature of (1.1), in which the data $\mathbf{p}$ and $\mathbf{b}$ of the lower-level linear programs are variables in the upper-level program, and the variables $\mathbf{q}^\star$ and $\boldsymbol{\pi}^\star$ of the upper-level program are constrained to be solutions of the lower-level linear programs.

By writing out the KKT optimality conditions of the linear program we can transform (1.1) into the following nonlinear program:

$$\begin{aligned} \underset{\mathbf{p},\mathbf{q},\mathbf{b},\boldsymbol{\pi},\mathbf{s}}{\text{maximize}} \;& f(\mathbf{p},\mathbf{q},\boldsymbol{\pi}) \\ \text{subject to } & A\mathbf{q} = \mathbf{b}, \quad A^T\boldsymbol{\pi} + \mathbf{s} = \mathbf{p} && (1.2) \\ & \mathbf{q}^T\mathbf{s} = 0 && (1.3) \\ & \mathbf{0} \le \mathbf{q}, \mathbf{s}. && (1.4) \end{aligned}$$

Constraints (1.2) and (1.4) require $\mathbf{q}, \boldsymbol{\pi}$ and $\mathbf{s}$ to be primal and dual feasible. Together constraints (1.3) and (1.4) form an equilibrium constraint, or a complementarity condition, that requires the product $q_i s_i = 0$ for all i. This equilibrium constraint makes the above program part of a special class of optimization problems called Mathematical Programs with Equilibrium Constraints (MPECs) [Z.Q96]. However, in an effort to be more precise we prefer the name Bilinear Bilevel Programs (BLBPs). In this work we attempt to solve these BLBPs by solving the associated nonlinear program.

In Section 2 we describe the market model, the mechanisms by which the market sets prices and dispatches electricity and how these can be formulated as a pair of primal and dual linear programs. In Section 3 we consider a single generation firm operating in a electrical power market and formulate a BLBP for the firm's offer into the market to maximize the firm's profit. In Section 4 we consider a small-scale electrical transmission network and formulate a similar BLBP for a vertically-integrated utility firm operating in this network. In Section 5 we construct a set of BLBPs arising from this electrical transmission network. In Section

6 we analyze the ability of two nonlinear solvers to solve this set of BLBPs. We conclude, in Section 7, by discussing the inherit difficulties in solving BLBPs.

2 The Pool Market Model

We consider an electricity market operating under the *pool market model* [CEA02]. In the pool market model, an overseeing entity, the *pool operator*, receives electricity transaction bids and offers from consumers and suppliers. Once all bids and offers have been received the pool operator determines the market clearing quantity (MCQ) and market clearing price (MCP) to maximize the social welfare. Electrical power is then dispatched from suppliers to consumers according to the market clearing price.

A *consumer* is an entity that wishes to purchase electrical power; a consumer represents a load in the power system network. An example of a consumer is a local power company which supplies electricity directly to homes and businesses. A *supplier* is an entity that produces electricity; a supplier represents a generator in the power system network. An example of a supplier is a company that owns and operates a hydroelectric power plant. Consumers and suppliers are connected to each other via the transmission system network.

A consumer places a *bid* to buy a quantity of electrical power, expressed in Megawatt hours (MWh), at a particular price per MWh. Similarly, a supplier places an *offer* to sell a quantity of power at a particular price. Both bids and offers are defined in terms of a price and maximum quantity pair (p \$/MWh, $\bar{q}$ MWh).

The current bids and offers in the pool market model are not public knowledge. A single consumer or supplier will know only its own bid or offer, not those of the other entities in the market. Only the pool operator has access to all of the bids and offers.

Consumers and suppliers may not only submit single price quantity pairs, but a set of price quantity pairs, with each pair corresponding to a different demand or generation resource. These set of price quantity pairs is called *bid* and *offer curves.* These pairs are arranged in increasing order of offer price for suppliers, and decreasing order of bid price for consumers. The pool operator aggregates these offer and bid curves to construct a single supply and demand curve for the entire market (Figure 2.1). The supply and demand curves $s(q)$ and $d(q)$ are piecewise constant functions defined as follows:

$$s(q) = \begin{cases} p_{j_1} & q \in [0, \bar{q}_{j_1}] \\ p_{j_2} & q \in (\bar{q}_{j_1}, \bar{q}_{j_2} + \bar{q}_{j_1}] \\ \vdots & \vdots \\ p_{j_{|\mathcal{S}|}} & q \in \left(\sum_{j \in \mathcal{S}} \bar{q}_j - \bar{q}_{j_{|\mathcal{S}|-1}}, \sum_{j \in \mathcal{S}} \bar{q}_j \right] \end{cases} \qquad d(q) = \begin{cases} p_{i_1} & q \in [0, \bar{q}_{i_1}] \\ p_{i_2} & q \in (\bar{q}_{i_1}, \bar{q}_{i_2} + \bar{q}_{i_1}] \\ \vdots & \vdots \\ p_{i_{|\mathcal{C}|}} & q \in \left(\sum_{i \in \mathcal{C}} \bar{q}_i - \bar{q}_{i_{|\mathcal{C}|-1}}, \sum_{i \in \mathcal{C}} \bar{q}_i \right] \end{cases}$$

Here $\mathcal{S}$ is the set of all suppliers and $\mathcal{C}$ is the set of all consumers, The indices $\{j_k\}_{k=1}^{|\mathcal{S}|}$ and $\{i_k\}_{k=1}^{|\mathcal{C}|}$ are defined by the price orderings $p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_{|\mathcal{S}|}}$, and $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_{|\mathcal{C}|}}$.

Once the supply and demand curves are constructed, the pool operator determines a single *market clearing quantity* (in MWh) and *market clearing price* (in \$/MWh). The market clearing quantity is the total amount of electric power transfered from suppliers to consumers. The market clearing price is the price per MWh charged for the transferred electric power. The MCQ and MCP are defined as the coordinates of the point of intersection of the supply and demand curves (Figure 2.1). Mathematically, the MCP and MCQ can be defined via the solution to a linear program that seeks to maximize the *social welfare.*

The social welfare is defined as the sum of the total *consumer profit* and the total *supplier profit.* Suppose that consumers and suppliers submit their respective bids and offers according to their operating costs. Then, the profit of an individual consumer i is $(p_i - \text{MCP})q_i$; since consumer i was willing to pay p_i \$/MWh for q_i MWh of electricity but instead only paid MCP \$/MWh. Similarly, the profit of an individual supplier j is $(\text{MCP} - p_j)q_j$; since supplier j received MCP \$/MWh for q_j MWh of electricity that cost only p_j \$/MWh to generate. Summing over all consumers and suppliers we arrive at the social welfare

$$\sum_{i\in\mathcal{C}}(p_i - \text{MCP})q_i + \sum_{j\in\mathcal{S}}(\text{MCP} - p_j)q_j = \sum_{i\in\mathcal{C}} p_i q_i - \sum_{j\in\mathcal{S}} p_j q_j + \text{MCP}\left(\sum_{j\in\mathcal{S}} q_j - \sum_{i\in C} q_j\right)$$

If we impose the additional constraint the supply must equal demand, or that $\sum_{i\in C} q_i = \sum_{j\in S} q_j$ we arrive at the following equation:

$$\text{social welfare} = \sum_{i\in\mathcal{C}} p_i q_i - \sum_{j\in\mathcal{S}} p_j q_j$$

Graphically, the social welfare corresponds to area between the demand and supply curves (shown hatched in Figure 2.1). This area is the sum of the consumer profit (the light gray region) and the supplier profit (the dark gray region).

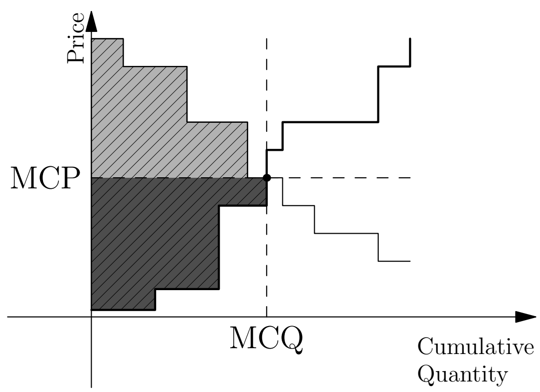

Figure 2.1: Supply (thick) and Demand (thin) curves for the pool market. The point of intersection represents the unique (MCQ, MCP).

To obtain the MCQ the pool operator solves the linear program

$$(P)\quad \begin{array}{lll} \max\limits_{\mathbf{q}_c,\mathbf{q}_s} & \mathbf{p}_c^T\mathbf{q}_c - \mathbf{p}_s^T\mathbf{q}_s & \\ \text{subject to} & \mathbf{e}^T\mathbf{q}_c - \mathbf{e}^T\mathbf{q}_s \le 0 & : \pi \\ & \mathbf{0} \le \mathbf{q}_c \le \bar{\mathbf{q}}_c & : \mathbf{z}_c \\ & \mathbf{0} \le \mathbf{q}_s \le \bar{\mathbf{q}}_s & : \mathbf{z}_s \end{array} \tag{2.1}$$

The objective of (P) is to maximize the social welfare. The primal decision variables $\mathbf{q}_c \in \mathbb{R}^{|\mathcal{C}|}$ and $\mathbf{q}_s \in \mathbb{R}^{|\mathcal{S}|}$ represent actual quantities of power exchanged and are portions of the maximum quantities of consumer bids and a supplier offers. Hence, the constraints $\mathbf{0} \leq \mathbf{q}_s \leq \bar{\mathbf{q}}_s$ and $\mathbf{0} \leq \mathbf{q}_c \leq \bar{\mathbf{q}}_c$. Here the upper bounds $\bar{\mathbf{q}}_c \in \mathbb{R}^{|\mathcal{C}|}$ and $\bar{\mathbf{q}}_s \in \mathbb{R}^{|\mathcal{S}|}$ are maximum quantity vectors; individual components of these vectors come from the maximum quantities, $\bar{q}$, specified in consumer bids and supplier offers. These upper bounds, $\bar{\mathbf{q}}_c$ and $\bar{\mathbf{q}}_s$, are thus data to the primal program. Here $\mathbf{p}_c \in \mathbb{R}^{|\mathcal{C}|}$ and $\mathbf{p}_s \in \mathbb{R}^{|\mathcal{S}|}$ are price vectors; individual components of these vectors correspond to consumer bid, and supplier offer, prices. Since the price p of a consumer's bid or a supplier's offer is fixed $\mathbf{p}_c$ and $\mathbf{p}_s$ are data to the primal program. This model of a fixed price but varying quantity is displayed graphically in the piecewise constant supply and demand curves shown in Figure 2.1.

We require that the market *clears*, or that the total quantity supplied is equal to the total demand. This is represented in the constraint $\mathbf{e}^T\mathbf{q}_c - \mathbf{e}^T\mathbf{q}_s = 0$, here $\mathbf{e}$ is a vector of all ones. This constraint requires that supply $\sum_{j \in \mathcal{S}} q_j$ equals demand $\sum_{i \in \mathcal{C}} q_i$. In (2.1) we have replaced $\mathbf{e}^T\mathbf{q}_c - \mathbf{e}^T\mathbf{q}_s = 0$ by the constraint $\mathbf{e}^T\mathbf{q}_c - \mathbf{e}^T\mathbf{q}_s \leq 0$; at the optimal solution this constraint will hold with equality. The inequality forces the Lagrange multiplier, π, associated with this constraint to be nonnegative. This is important, for, as we shall see later, π represents a price.

From the solution to the primal program, $\mathbf{q}_c^\star$ and $\mathbf{q}_s^\star$, it is possible to determine the market clearing quantity, since $\text{MCQ} = \sum_{j \in \mathcal{S}} q_j^\star = \mathbf{e}^T\mathbf{q}_s^\star = \mathbf{e}^T\mathbf{q}_c^\star$.

To determine the market clearing price we turn to (D) the dual of (P).

$$(D) \qquad \begin{array}{lrl} \min\limits_{\pi,\mathbf{z}_c,\mathbf{z}_s} & \bar{\mathbf{q}}_c^T\mathbf{z}_c + \bar{\mathbf{q}}_s^T\mathbf{z}_s & \\ \text{subject to} & \pi\mathbf{e} + \mathbf{z}_c \geq \ \ \mathbf{p}_c & : \mathbf{q}_c \\ & -\pi\mathbf{e} + \mathbf{z}_s \geq -\mathbf{p}_s & : \mathbf{q}_s \\ & \mathbf{0} \leq \mathbf{z}_c, \mathbf{z}_s, \pi & \end{array} \tag{2.2}$$

Let $\pi^\star, \mathbf{z}_c^\star, \mathbf{z}_s^\star$ be an optimal solution to (D). Then $\pi^\star \in \mathbb{R}$ is a market clearing price, while $\mathbf{z}_c^\star \in \mathbb{R}^{|\mathcal{C}|}$ and $\mathbf{z}_s^\star \in \mathbb{R}^{|\mathcal{S}|}$ represent the shadow prices associated with the maximum quantity constraints $\mathbf{q}_c \leq \bar{\mathbf{q}}_c$ and $\mathbf{q}_s \leq \bar{\mathbf{q}}_s$ respectively.

When the supply and demand curves intersect at a single point the MCQ and MCP are uniquely determined (this can be seen in Figure 2.2a). When this occurs the solutions to the primal and dual linear programs are unique. If, however, the supply and demand curves intersect at multiple points, then either the MCQ or the MCP is not uniquely determined (as can be seen in Figure 2.2b-c); this corresponds to multiple solutions in the primal or the dual linear program. In these cases the pool operator must choose a unique (MCQ,MCP) pair. When the MCQ can take on a range of values the pool operator chooses the MCQ to maximize the quantity sold (Figure 2.2b). When the MCP can take on a range of values the pool operator chooses the MCP to minimize price (Figure 2.2c).

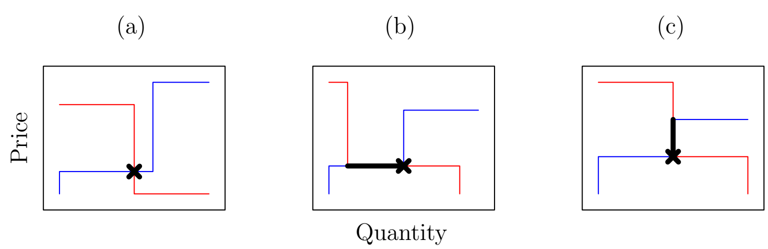

Figure 2.2: The three different types of supply (blue) and demand (red) curve intersections. The set of acceptable (MCQ,MCP) pairs is shown in black, the chosen solution is crossed.

Once the MCQ $= \mathbf{e}^T\mathbf{q}_c^\star$ and MCP $= \pi^\star$ have been determined, consumers pay a total

(MCP · MCQ) $= \pi^\star \mathbf{e}^T \mathbf{q}_c^\star$ dollars and receive a total of MCQ MWh of power. If consumer $i \in \mathcal{C}$ placed a bid $(p_i, \bar{q}_i)$ whose corresponding quantity $q_i^\star > 0$, then consumer i pays a total of $\pi^\star q_i^\star$ dollars and receives $q_i^\star$ MWh of power. If supplier $j \in \mathcal{S}$ placed an offer $(p_j, \bar{q}_j)$ whose corresponding quantity $q_j^\star > 0$, then supplier j receives $\pi^\star q_j^\star$ dollars and supplies $q_j^\star$ MWh of power.

3 The Single Supplier Problem

We have discussed how prices are set and electricity is dispatched, in a pool market, given a set of bids and offers. These bids and offers are placed into the market by individual consumers and suppliers. We refer to an individual consumer or supplier with control over a single bid or offer as a market *entity*. Each market entity faces the problem of choosing its bid or offer to maximize its profit.

This problem can be modeled as an optimization problem where we remove the entity from the market, with all other consumer bids and supplier offers fixed, and choose that entity's price max quantity pair $(p_k, \bar{q}_k)$ to optimize its profit. Recall that the pool operator chooses the MCQ and MCP to maximize the social welfare based on bids and offers submitted by consumers and suppliers, assuming that these bids and offers represent true marginal costs. However, consumers and suppliers are allowed to choose their own bids and offers. By varying its bid or offer from marginal cost a firm may be able to increase its profit [HMP00].

Consider a single supplier $k \in \mathcal{S}$, and let c_k, in \$/MWh, represent the true marginal cost of supplier k. That is, it costs supplier k, $c_k q$ dollars to generate q MWh of power. We will assume that there exists a quantity of power, $q_k^{\max}$, that is an upper bound on the total amount of power supplier k can produce. This upper bound arises from the physical limitations of power generation; for example it may be impossible to generate more than

$q_k^{\max}$ with a supplier's nuclear power plant.

The supplier wishes to maximize profit, or the revenue (MCP $\cdot\, q_k^\star$) $= \pi^\star q_k^\star$ minus the cost $c_k q_k^\star$. Here we see that the profit of supplier k depends on the solutions to (2.1) and (2.2). In particular, it depends on the MCP, $\pi^\star$, and the actual quantity, $q_k^\star$, that the pool operator requests supplier k to produce. These values in turn depend on the bids and offers placed by other consumers and suppliers. Unfortunately, these bids and offers are not known to supplier k. For this work, however, we will assume that supplier k knows, or is able to produce an estimate of, the other consumer and supplier bids and offers. That is, we assume that the price and max quantity vectors $\mathbf{p}_s, \bar{\mathbf{q}}_s \in \mathbb{R}^{|\mathcal{S}|-1}$ of the other suppliers and the price and max quantity vectors $\mathbf{p}_c, \bar{\mathbf{q}}_c \in \mathbb{R}^{|\mathcal{C}|}$ of the consumers are known.

With this information, the Single Supplier Problem is to optimize the profit of supplier k subject to the constraint that the market is in equilibrium. The equilibrium constraint requires us to incorporate the optimality conditions of the pool market primal and dual linear programs. This yields the following problem:

$$
\begin{array}{rll}
\underset{p_k,\bar{q}_k,q_k\mathbf{q}_c,\mathbf{q}_s,\pi,\mathbf{z}_c,\mathbf{z}_s,\xi}{\text{maximize}} & \pi q_k - c_k q_k & \\
\text{subject to} & \mathbf{e}^T\mathbf{q}_c - \mathbf{e}^T\mathbf{q}_s - q_k \le 0 & \\
& 0 \le q_k \le \bar{q}_k, & -\pi \;\; + \xi \;\ge -p_k \\
& 0 \le \mathbf{q}_c \le \bar{\mathbf{q}}_c, & \pi\mathbf{e} + \mathbf{z}_c \ge \quad \mathbf{p}_c \\
& 0 \le \mathbf{q}_s \le \bar{\mathbf{q}}_s, & -\pi\mathbf{e} + \mathbf{z}_s \ge -\mathbf{p}_s \\
& 0 \le \bar{q}_k \le q_k^{\max} & \mathbf{z}_c, \mathbf{z}_s, \xi, \pi \ge 0 \\
& \mathbf{p}_c^T\mathbf{q}_c - \mathbf{p}_s^T\mathbf{q}_s - p_k q_k \ge & \bar{\mathbf{q}}_c^T\mathbf{z}_c + \bar{\mathbf{q}}_s^T\mathbf{z}_s + \bar{q}_k\xi
\end{array}
\tag{3.1}
$$

Here $\xi \in \mathbb{R}$ is the Lagrange multiplier or dual variable associated with the constraint $q_k \le \bar{q}_k$. Note that $p_k, \bar{q}_k \in \mathbb{R}$ are decision variables and thus supplier k's offer $(p_k, \bar{q}_k)$ is to be determined. The marginal cost of generating power, c_k, and the maximum generation capacity, $q_k^{\max}$, are data to the problem along with the vectors, $\mathbf{p}_c, \mathbf{p}_s, \bar{\mathbf{q}}_c$ and $\bar{\mathbf{q}}_s$.

The presence of the term πq_k makes the objective function bilinear. Note that this a bilinear bilevel program. In fact, this is the simplest bilinear bilevel program arising from the electric power market model.

A graphical representation of Problem (3.1) is shown in Figure 3.1. The residual demand curve is constructed by grouping together all other consumers and suppliers into a single entity that has a demand for supplier k's power. For supply to equal demand supplier k must produce a quantity of electrical power at a price on this residual demand curve. Different points on this residual demand curve will yield different amounts of profit. The point which yields the maximum profit is the optimal offer $(p_k^\star, \bar{q}_k^\star)$ for supplier k.

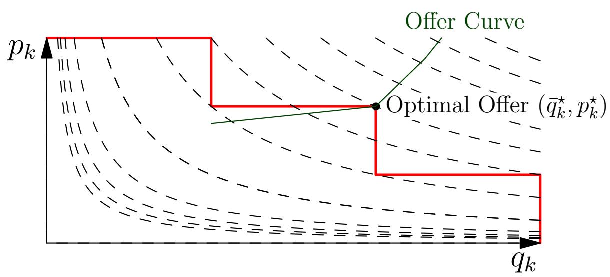

Figure 3.1: Residual demand curve (red) and lines of isoprofit (dashed) for the BLBP.

The solution of the Single Supplier Problem yields the optimal offer $(p_k^\star, \bar{q}_k^\star)$ for supplier k. Unfortunately, this is a single point solution. Supplier's often submit a set of price max quantity pairs into the pool market. These sets of offers give rise to offer curves. Let $o_k(\cdot)$ denote the offer curve for supplier k. Any offer curve o_k which satisfies $o_k(\bar{q}_k^\star) = p_k^\star$ will yield the optimal profit for supplier k. However, different offer curves can yield different amounts of profit, especially in the presence of uncertainty regarding the other firms' bids and offers. The choice of an offer curve $o_k(\cdot)$ that minimizes risk is performed separately after solving the Single Supplier Problem, and is beyond the scope of this work.

4 The Transmission Incentive Problem

4.1 The Two-Node Network Market Model

We now consider a problem where the structure of the transmission network comes into play [Ent07]. A two-node network is shown in Figure 4.1, where each node represents a different physical location. There are four separate entities in the market: Supplier 1 and Demand 1 located at Node 1, and Supplier 2 and Demand 2 located at Node 2. Demand 1 and Demand 2 receive payment from customers at nodes 1 and 2 and place bids into the pool market to buy electrical power to supply to their customers. Supplier 1 and Supplier 2 generate electricity and place offers into the pool market to sell electrical power. In addition, nodes 1 and 2 are connected together by means of a transmission line.

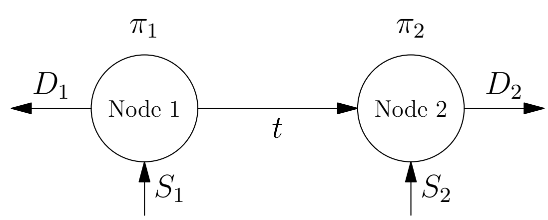

Figure 4.1: A diagram of the two-node network.

Although this two-node network may seem overly simple it actually models several important dynamics arising in practice. Historically electric power generators were built close to city centers (in this model Node 2)[JT00]. However, as generators became larger, urban sites became scarce. This, coupled with improvements in transmission technology, caused newer, more efficient generators, to be sited away from heavy load centers[JT00]. In this model the more efficient generators are located at Node 1. There are numerous examples of electric power networks, and sub-networks in California, England, Argentina, Chile, and New Zealand that have similar properties to this two-node network model [JT00].

Since nodes 1 and 2 are physically separated we will treat the electrical power at each of these nodes as separate products. The price of electricity at Node 1 is π_1 \$/MWh and the price of electricity at Node 2 is π_2 \$/MWh. Each of the suppliers generate enough electricity to meet the demand of the local market, however, Supplier 1 has the ability to generate electricity at lower prices than Supplier 2. Neither supplier has enough capacity to meet the demand of the entire market. Since Supplier 1 can generate cheaper electricity than Supplier 2, we expect that $\pi_1 \leq \pi_2$. Thus, it might be advantageous to move energy from Node 1 to Node 2 along the transmission line.

We will denote the quantity of energy that flows across this transmission line as t and assume there is a maximum transmission capacity of $\bar{t}$. The transmission line is uni-directional; energy can only flow from Node 1 to Node 2. The owner of the transmission line receives a congestion fee for transmitting t MWh across the transmission line. The fee is based on the differences in the prices π_1 and π_2 and is $(\pi_2 - \pi_1)t$ dollars. In this model we will assume that there is no cost to the transmission line owner to transmit energy, and that there is no power loss due to transmission.

Demand 1 and Demand 2 each submit bids $(p_{D_k}, \bar{q}_{D_k}), k = 1, 2$ into the market. In this model Supplier 1 and Supplier 2 both control multiple generation resources, and thus will submit multiple offers into the market.

Recall that the social welfare is the sum of total consumer profit and the total supplier profit. The total consumer profit is

$$(p_{D_1} - \pi_1)q_{D_1} + (p_{D_2} - \pi_2)q_{D_2}$$

and the total supplier profit due to generation is

$$\sum_{i \in \mathcal{S}_1} (\pi_1 - p_i)q_i + \sum_{i \in \mathcal{S}_2} (\pi_2 - p_i)q_i.$$

Here $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets that index the generation resources controlled by Supplier 1 and Supplier 2.

We can also think of the transmission line owner as supplier of transmission. Since there is no cost to produce transmission, the transmission supplier's profit is just its revenue $(\pi_2 - \pi_1)t$.

Thus, the social welfare is given by

$$(p_{D_1} - \pi_1)q_{D_1} + (p_{D_2} - \pi_2)q_{D_2} + \sum_{i \in \mathcal{S}_1}(\pi_1 - p_i)q_i + \sum_{i \in \mathcal{S}_2}(\pi_2 - p_i)q_i + (\pi_2 - \pi_1)t$$

Rearranging this we have that

$$\begin{aligned}\text{social welfare} = &\, p_{D_1}q_{D_1} + p_{D_2}q_{D_2} - \sum_{i \in \mathcal{S}_1} p_i q_i - \sum_{i \in \mathcal{S}_2} p_i q_i \\ &+ \pi_1 \left(\sum_{i \in \mathcal{S}_1} q_i - q_{D_1} - t \right) + \pi_2 \left(\sum_{i \in \mathcal{S}_2} q_i - q_{D_2} + t \right).\end{aligned}$$

When we require that supply equal demand at each of the nodes, or that

$$\sum_{i \in \mathcal{S}_1} q_i - t = q_{D_1} \quad \text{and} \quad \sum_{i \in \mathcal{S}_2} q_i + t = q_{D_2},$$

we arrive at the following equation for the social welfare

$$\text{social welfare} = p_{D_1}q_{D_1} + p_{D_2}q_{D_2} - \sum_{i \in \mathcal{S}_1} p_i q_i - \sum_{i \in \mathcal{S}_2} p_i q_i.$$

Thus, the market clearing linear program has the form

$$\begin{array}{lll} \underset{q_{D_1}, q_{D_2}, \mathbf{q}_{S_1}, \mathbf{q}_{S_2}, t}{\text{maximize}} & p_{D_1}q_{D_1} + p_{D_2}q_{D_2} - \mathbf{p}_{S_1}^T \mathbf{q}_{S_1} - \mathbf{p}_{S_2}^T \mathbf{q}_{S_2} & \\ \text{subject to} & -\mathbf{e}^T \mathbf{q}_{S_1} + t + q_{D_1} \leq 0 & : \pi_1 \\ & -\mathbf{e}^T \mathbf{q}_{S_2} - t + q_{D_2} \leq 0 & : \pi_2 \\ & 0 \leq q_{D_1} \leq \bar{q}_{D_1} & : \xi_{D_1} \\ & 0 \leq q_{D_2} \leq \bar{q}_{D_2} & : \xi_{D_2} \\ & \mathbf{0} \leq \mathbf{q}_{S_1} \leq \bar{\mathbf{q}}_{S_1} & : \mathbf{z}_{S_1} \\ & \mathbf{0} \leq \mathbf{q}_{S_2} \leq \bar{\mathbf{q}}_{S_2} & : \mathbf{z}_{S_2} \\ & 0 \leq t \quad \leq \bar{t} & : \pi_t \end{array} \tag{4.1}$$

The objective of this linear program is to maximize the social welfare. Here $\mathbf{p}_{S_1}, \bar{\mathbf{q}}_{S_1} \in \mathbb{R}^{|\mathcal{S}_1|}$ and $\mathbf{p}_{S_2}, \bar{\mathbf{q}}_{S_2} \in \mathbb{R}^{|\mathcal{S}_2|}$ are the price and max quantity vectors for Supplier 1 and Supplier 2 respectively; the individual components of these vectors are the price max quantity pairs $(p_i, \bar{q}_i)$. These vectors are the problem data.

Note that we have transformed the two equality constraints, which ensure supply equals demand at each node,

$$\left.\begin{aligned} \mathbf{e}^T\mathbf{q}_{S_1} - t - q_{D_1} &= 0 \\ \mathbf{e}^T\mathbf{q}_{S_2} + t - q_{D_2} &= 0 \end{aligned}\right\} \quad \text{into} \quad \left\{\begin{aligned} -\mathbf{e}^T\mathbf{q}_{S_1} + t + q_{D_1} &\leq 0 \\ -\mathbf{e}^T\mathbf{q}_{S_2} - t + q_{D_2} &\leq 0 \end{aligned}\right. .$$

This transformation ensures that the dual variables associated with these constraint, π_1 and π_2, are nonnegative. This is important as π_1 and π_2 are the market clearing prices at Node 1 and Node 2.

The dual of (4.1) is then

$$\begin{array}{rrll} \underset{\pi_1,\pi_2,\xi_{D_1},\xi_{D_2},\mathbf{z}_{S_1},\mathbf{z}_{S_2},\pi_t}{\text{minimize}} & \bar{q}_{D_1}\xi_{D_1} + \bar{q}_{D_2}\xi_{D_2} & +\bar{\mathbf{q}}_{S_1}^T\mathbf{z}_{S_1} + \bar{\mathbf{q}}_{S_2}^T\mathbf{z}_{S_2} + \bar{t}\pi_t & \\ \text{subject to} & \pi_1 + \xi_{D_1} & \geq p_{D_1} & : q_{D_1} \\ & \pi_2 + \xi_{D_2} & \geq p_{D_2} & : q_{D_2} \\ & -\pi_1\mathbf{e} + \mathbf{z}_{S_1} & \geq -\mathbf{p}_{S_1} & : \mathbf{q}_{S_1} \\ & -\pi_2\mathbf{e} + \mathbf{z}_{S_2} & \geq -\mathbf{p}_{S_2} & : \mathbf{q}_{S_2} \\ & \pi_t & \geq \pi_2 - \pi_1 & : t \\ & \pi_1, \pi_2 & \geq 0 & \\ & \xi_{D_1}, \xi_{D_2}, \mathbf{z}_{S_1}, \mathbf{z}_{S_2}, \pi_t & \geq 0 & \end{array} \tag{4.2}$$

Here, again, the variables $\bar{q}_{D_1}, \bar{q}_{D_2} \in \mathbb{R}, \bar{\mathbf{q}}_{S_1} \in \mathbb{R}^{|\mathcal{S}_1|}, \bar{\mathbf{q}}_{S_2} \in \mathbb{R}^{|\mathcal{S}_2|}$ and $\bar{t} \in \mathbb{R}$ represent maximum quantities of electrical power. These quantities, along with the prices $p_{D_1}, p_{D_2} \in \mathbb{R}$ and $\mathbf{p}_{S_1} \in \mathbb{R}^{|\mathcal{S}|_1}, \mathbf{p}_{S_2} \in \mathbb{R}^{|\mathcal{S}|_2}$, are problem data.

The variables ξ_{D_1} and ξ_{D_2} are the shadow prices for Demand 1 and Demand 2. Suppose that the bid of Demand 1 is partially rejected, and so we have $q_{D_1} \neq \bar{q}_{D_1}$. This implies that the dual variable $\xi_{D_1} = 0$, and thus the constraint $\pi_1 + \xi_{D_1} \geq p_{D_1}$ becomes $\pi_1 \geq p_{D_1}$. Note that this is when we would reject a bid; when the bid price $p_{D_1} < \pi_1$. Thus the constraints $\pi_1 + \xi_{D_1} \geq p_{D_1}$ and $\pi_2 + \xi_{D_2} \geq p_{D_2}$ ensure that for those demand bids accepted $\pi_i \leq p_{D_i}$.

Similarly, the variables $\mathbf{z}_{S_1} \in \mathbb{R}^{|\mathcal{S}_1|}$ and $\mathbf{z}_{S_2} \in \mathbb{R}^{|\mathcal{S}_2|}$ are the shadow prices for Supplier 1 and Supplier 2 on the constraints $\mathbf{q}_{S_1} \leq \bar{\mathbf{q}}_{S_1}$ and $\mathbf{q}_{S_2} \leq \bar{\mathbf{q}}_{S_2}$. Suppose that the offer of Supplier 1 was rejected, and so we have $\mathbf{q}_{S_1} \neq \bar{\mathbf{q}}_{S_1}$. This implies that the dual variable $\mathbf{z}_{S_1} = 0$, and thus the constraint $-\pi_1\mathbf{e} + \mathbf{z}_{S_1} \geq -\mathbf{p}_{S_1}$ becomes $-\pi_1\mathbf{e} \geq -\mathbf{p}_{S_1}$ or $\pi_1\mathbf{e} \leq \mathbf{p}_{S_1}$. Note that this is when we would reject an offer; when the offer price $\mathbf{p}_{S_1} > \pi_1\mathbf{e}$. Thus the constraints $-\pi_1\mathbf{e} + \mathbf{z}_{S_1} \geq -\mathbf{p}_{S_1}$ and $-\pi_2\mathbf{e} + \mathbf{z}_{S_2} \geq -\mathbf{p}_{S_2}$ ensure that for those supplier offers accepted $\mathbf{p}_{S_i} \leq \pi_i\mathbf{e}$. Here, for ease of notation, we have used vector inequalities to describe a collection of scalar inequalities.

The variable π_t is the price of transmission. To see this, note that $\pi_t \geq \pi_2 - \pi_1$. But, since $\pi_t \geq 0$ and $\bar{t} \geq 0$, and part of the dual objective is to minimize the quantity $\bar{t}\pi_t$, the constraint will be on its bound, with $\pi_t = \pi_2 - \pi_1$. This is exactly the marginal price charged by the transmission line owner for transmission.

4.1.1 Experimental Setup

So far the market model of the two-node network has been reasonably general. We have not specified any values for quantities of electricity or for marginal prices in the market—other than to say that we expect the price of electricity at Node 1 will be less than that at Node 2. Now we specify a particular instance of the two-node network by providing concrete values for the cost and capacity of the generation resources and the revenue and quantity of the consumer demand.

There are a total of 5 power generating resources: three located at Node 1, and two located at Node 2. Table 4.1 shows the marginal cost of operation, and the maximum generation capacity of these resources. Note that there are more generating resources at Node 1, and for each resource at Node 1, the cost of generating electricity is the same or less than the corresponding resources at Node 2.

Table 4.1: Marginal cost and maximum capacity data for generating resources.

Supplier 1 (at Node 1)	Marginal Cost ($/MWh)	Maximum Capacity (MWh)
Resource 1	25	3000
Resource 2	25	3000
Resource 3	35	3000
Supplier 2 (at Node 2)	Marginal Cost ($/MWh)	Maximum Capacity (MWh)
Resource 4	25	3000
Resource 5	36	3000

Both Demand 1 and Demand 2 charge their customers a fixed rate of 100 $/MWh for electric power, and their customers demand 5000 MWh. Finally, the maximum transmission capacity is 2.5 GWh. It should be noted that this quantity is larger than maximum generation capacity at either of the two nodes; so the transmission link should never be congested, or saturated, as a result of physical limitations.

4.2 Marginal Cost Market

We now perform the action of a pool operator in the two-node network and calculate the market clearing prices and market clearing quantities in order to dispatch electricity. To do this we need to set values for the bids and offers of Demand 1, Demand 2, Supplier 1 and Supplier 2. We will have entities in the market set their bids and offers at the marginal cost (or revenue) and maximum quantity available.

Table 4.2 shows the values of the bid and offer data into (4.1) and (4.2), as well as the optimal quantities computed, and the profit earned by each entity in the market.

As expected, the cheaper generation resources of Supplier 1 at Node 1 are dispatched before the more expensive resources of Supplier 2 at Node 2. A total of 7000 MWh are dispatched from Supplier 1, 5000 MWh of these go to meet the demand at Node 1, the other 2000 MWh are transmitted to Node 2. These 2000 MWh, along with the 3000 MWh from the cheapest resource of Supplier 2, meet the 5000 MWh demand at Node 2.

Since the transmitted quantity of 2000 MWh was well under the maximum transmission capacity of 2.5 GWh the market clearing prices at each node are the same; namely 35 $/MWh. This amount corresponds to the price of the most expensive generation resource dispatched.

Because the transmission line is not congested, and thus the market clearing price at Node 1 and Node 2 are the same, the transmission line owner earns no revenue and makes no profit. Since the market clearing price is small compared to the price Demand 1 and Demand 2 charge their customers, these entities accrue the most profit. However, Supplier 1 also makes profit on its two inexpensive generation resources.

Table 4.2: Marginal Cost Market Data and Results

Bid and Offer Input Data

Demand 1 Bid	$(p_{D_1}, \bar{q}_{D_1}) = (5000 \text{ MWh}, 100 \text{ \$/MWh})$
Demand 2 Bid	$(p_{D_2}, \bar{q}_{D_2}) = (5000 \text{ MWh}, 100 \text{ \$/MWh})$
Supplier 1 Offer	$(\mathbf{p}_{S_1}, \bar{\mathbf{q}}_{S_1}) = \left(\begin{bmatrix} 25 \\ 25 \\ 35 \end{bmatrix} \text{\$/MWh}, \begin{bmatrix} 3000 \\ 3000 \\ 3000 \end{bmatrix} \text{MWh} \right)$
Supplier 2 Offer	$(\mathbf{p}_{S_2}, \bar{\mathbf{q}}_{S_2}) = \left(\begin{bmatrix} 25 \\ 36 \end{bmatrix} \text{\$/MWh}, \begin{bmatrix} 3000 \\ 3000 \end{bmatrix} \text{MWh} \right)$
Transmission Maximum Capacity	$\bar{t} = 2.5\text{GWh}$

Results

Quantity Demand 1	$q_{D_1} = 5000$ MWh
Quantity Demand 2	$q_{D_2} = 5000$ MWh
Quantity Supply 1	$\mathbf{q}_{S_1}^T = \begin{bmatrix} 3000 & 3000 & 1000 \end{bmatrix}$ MWh
Quantity Supply 2	$\mathbf{q}_{S_2}^T = \begin{bmatrix} 3000 & 0 \end{bmatrix}$ MWh
Transmitted Quantity	$t = 2000$ MWh
MCP Node 1	$\pi_1 = 35$ \$/MWh
MCP Node 2	$\pi_2 = 35$ \$/MWh
Shadow Price Demand 1	$\xi_{D_1} = 65$ \$/MWh
Shadow Price Demand 2	$\xi_{D_2} = 65$ \$/MWh
Shadow Price Supply 1	$\mathbf{z}_{S_1}^T = \begin{bmatrix} 10 & 10 & 0 \end{bmatrix}$ \$/MWh
Shadow Price Supply 2	$\mathbf{z}_{S_2}^T = \begin{bmatrix} 10 & 0 \end{bmatrix}$ \$/MWh

Profit

Profit Transmission Owner	0 \$
Profit Demand 1	325,000 \$
Profit Demand 2	325,000 \$
Profit Supplier 1	60,000 \$
Profit Supplier 2	30,000 \$

4.3 The Two-Node Network Single Firm Problem

So far we have formulated the linear programs used by a pool operator to determine the market clearing prices, the market clearing quantities, and dispatch electricity in this two-node network. We now analyze a scenario where a vertically-integrated utility firm operates within this simple two-node network. Here a *firm* is single agent that controls a collection of market entities. We will analyze the effect on the market when this vertically-integrated firm seeks to maximize its profit.

Let Firm 2 be a vertically-integrated utility located at Node 2. Firm 2 owns and operates Supplier 2, Demand 2, and the transmission line. Therefore it has three sources of revenue: revenue received from the generation and sale of electricity from Supplier 2, revenue received from transmission line fees, and revenue received from the sale of electricity to Demand 2. Firm 2 thus has the ability to set the prices (and maximum quantities) of the offer of Supplier 2 and the bid of Demand 2 into the market. In addition—and this will play a crucial role—Firm 2 has the ability to set, or control, the maximum transmission capacity of the transmission line.

The goal of Firm 2 is to maximize its total profit. To maximize Firm 2's profit we will construct a BLBP (similar to that discussed in Section 3, but with the structure of the two-node network taken into account) where the crucial decision variables are the bids, offers, and transmission capacities of the entities owned by Firm 2. We will assume, as we did in Section 3, that the bids and offers of the other entities in the market are known to Firm 2 and are thus data into the BLBP.

The first step in constructing the BLBP is to formulate the objective function. The profit of Firm 2 is given by

$$\pi_2 \mathbf{e}^T \mathbf{q}_{S_2} - \mathbf{c}_{S_2}^T \mathbf{q}_{S_2} + (\pi_2 - \pi_1)t + (r_{D_2} - \pi_2)q_{D_2}$$

Here $\mathbf{c}_{S_2} \in \mathbb{R}^{|S_2|}$ is a vector whose entries correspond to the operating costs of Supplier 2's generation resources, and $r_{D_2} \in \mathbb{R}$ is the revenue received by Demand 2 from its end customers. The objective of maximizing Firm 2's profit, along with the constraints from (4.1) and (4.2), yields the following BLBP:

$$\begin{array}{rlrl}
\underset{\substack{p_{D_1},\bar{q}_{D_1},\mathbf{p}_{S_2},\bar{\mathbf{q}}_{S_2},\bar{t},\\ q_{D_1},q_{D_2},\mathbf{q}_{S_1},\mathbf{q}_{S_2},t,\\ \pi_1,\pi_2,\pi_t,\xi_{D_1},\xi_{D_2},\mathbf{z}_{S_1},\mathbf{z}_{S_2}}}{\text{maximize}} & \multicolumn{3}{l}{(\pi_2\mathbf{e}^T - \mathbf{c}_{S_2}^T)\mathbf{q}_{S_2} + (\pi_2 - \pi_1)t + (r_{D_2} - \pi_2)q_{D_2}} \\
\text{subject to} & & & \\
-\mathbf{e}^T\mathbf{q}_{S_1} + t + q_{D_1} & \leq 0 & \pi_1 + \xi_{D_1} & \geq p_{D_1} \\
-\mathbf{e}^T\mathbf{q}_{S_2} - t + q_{D_2} & \leq 0 & \pi_2 + \xi_{D_2} & \geq p_{D_2} \\
0 \leq q_{D_1} & \leq \bar{q}_{D_1}, & -\pi_1 e + \mathbf{z}_{S_1} & \geq -\mathbf{p}_{S_1} \\
0 \leq q_{D_2} & \leq \bar{q}_{D_2}, & -\pi_2 e + \mathbf{z}_{S_2} & \geq -\mathbf{p}_{S_2} \\
\mathbf{0} \leq \mathbf{q}_{S_1} & \leq \bar{\mathbf{q}}_{S_1}, & \pi_t & \geq \pi_2 - \pi_1 \\
\mathbf{0} \leq \mathbf{q}_{S_2} & \leq \bar{\mathbf{q}}_{S_2}, & \pi_1, \pi_2 & \geq 0 \\
0 \leq t & \leq \bar{t}, & \xi_{D_1}, \xi_{D_2}, \mathbf{z}_{S_1}, \mathbf{z}_{S_2}, \pi_t & \geq 0 \\
\multicolumn{4}{c}{p_{D_1}q_{D_1} + p_{D_2}q_{D_2} - \mathbf{p}_{S_1}^T\mathbf{q}_{S_1} - \mathbf{p}_{S_2}^T\mathbf{q}_{S_2} \;\geq\; \bar{q}_{D_1}\xi_{D_1} + \bar{q}_{D_2}\xi_{D_2} + \bar{\mathbf{q}}_{S_1}^T\mathbf{z}_{S_1} + \bar{\mathbf{q}}_{S_2}^T\mathbf{z}_{S_2} + \bar{t}\pi_t} \\
0 \leq p_{D_2} & \leq r_{D_2} & \bar{q}_{D_2} & \geq q_{D_2}^{\min} \\
\mathbf{0} \leq \mathbf{c}_{S_2} & \leq \mathbf{p}_{S_2} & \mathbf{0} \leq \bar{\mathbf{q}}_{S_2} & \leq \mathbf{q}_{S_2}^{\max} \\
0 \leq \bar{t} & \leq t^{\max} & &
\end{array} \tag{4.3}$$

The key decision variables in this problem are the bids, offers, and transmission capacities, controlled by Firm 2; these are: Demand 2's bid $(p_{D_2}, \bar{q}_{D_2})$, Supplier 2's offer $(\mathbf{p}_{S_2}, \bar{\mathbf{q}}_{S_2})$, and the maximum transmission capacity $\bar{t}$. The optimal values of these variables will determine what Firm 2 gives to the pool operator. The data for this problem are the bids and offers of the other entities in the market: Demand 1's bid $(p_{D_1}, \bar{q}_{D_1})$, and Supplier 1's offer $(\mathbf{p}_{S_1}, \bar{\mathbf{q}}_{S_1})$. Secondary variables for this problem, which arise from including the primal and dual linear program, are: the market quantities $q_{D_1}, q_{D_2}, \mathbf{q}_{S_1}, \mathbf{q}_{S_2}$, the transmitted quantity t, the price of transmission π_t, the market clearing price at each node π_1 and π_2, and the shadow prices $\xi_{D_1}, \xi_{D_2}, \mathbf{z}_{S_1}$, and $\mathbf{z}_{S_2}$.

Because the bid, offer, and maximum transmission capacity are decision variables in the BLBP we need additional constraints and data to ensure that these variables do not violate physical limitations. The maximum transmission capacity, $\bar{t}$, that Firm 2 reports to the pool operator must not exceed the physical capacity of the line, $t^{\max}$, which is 2.5 GWh. The price of Supplier 2's offer into the market, $\mathbf{p}_{S_2}$, should be more than the operating cost of its generation resources, $\mathbf{c}_{S_2}$. The maximum generation quantity of Supplier 2's offer into the market, $\bar{\mathbf{q}}_{S_2}$, must not exceed the maximum physical generation capacity of its generation resources, $\mathbf{q}_{S_2}^{\max}$. The price of Demand 2's bid into the market, p_{D_2}, should not exceed the revenue generated from customer sales, r_{D_2}. The maximum quantity of Demand 2's bid, $\bar{q}_{D_2}$, must meet or exceed the minimum quantity of electricity demanded by its customers, $q_{D_2}^{\min}$.

4.4 Experimental Results for the Two-Node Network Single Firm Problem

With the formulation complete, we now turn to numerically computing the bid, offer, and maximum transmission quantities that optimize Firm 2's profit.

To do this we must first specify the bids and offers of the other entities into the market—here we have all other entities offer (bid) marginal cost (revenue) into the market at the maximum quantity. In addition, we must specify the physical data associated with the problem. Table 4.3 specifies the values of all data in the BLBP.

Table 4.3: Input data for Two-Node Network Single Firm BLBP.

Bids and Offers

Demand 1 Bid	$(p_{D_1}, \bar{q}_{D_1}) = (100\ \$/\text{MWh}, 5000\ \text{MWh})$
Supplier 1 Offer	$(\mathbf{p}_{S_1}, \bar{\mathbf{q}}_{S_1}) = \left(\begin{bmatrix} 25 \\ 25 \\ 35 \end{bmatrix} \$/\text{MWh}, \begin{bmatrix} 3000 \\ 3000 \\ 3000 \end{bmatrix} \text{MWh} \right)$

Physical Data

Maximum Physical Transmission Capacity	$t^{\max} = 2.5$ GWh
Supplier 2 Operating Cost	$\mathbf{c}_{S_2}^T = \begin{bmatrix} 25 & 36 \end{bmatrix}$ \$/MWh
Supplier 2 Maximum Generation Capacity	$\mathbf{q}^{\max T}_{S_2} = \begin{bmatrix} 3000 & 3000 \end{bmatrix}$ MWh
Demand 2 Consumer Revenue	$r_{D_2} = 100$ \$/MWh
Demand 2 Minimum Consumer Demand	$q_{D_2}^{\min} = 5000$ MWh

With the problem completed specified, we use the fully nonlinear solver SNOPT [GMS97] to solve the nonlinear formulation of the BLBP. SNOPT was designed to handle large-scale nonlinear optimization problems with a large number of nonlinear constraints. This problem, with merely 24 variables and 15 constraints (of these constraints, only a single is nonlinear), is considered small. As expected, SNOPT quickly converges to a local, and in this case the global, optimum, performing only 39 minor iterations and 15 function evaluations.

Table 4.4 shows the optimal bid, offer, and maximum transmission quantity, computed by SNOPT, as well as the profit for Firm 2 and the individual entities in the market.

These results are very interesting. The first value to notice is the maximum transmission quantity, $\bar{t} = 1000$ MWh. Although, the transmission line has a maximum physical capacity of 2.5 GWh, Firm 2 has chosen to report a capacity of merely 1000 MWh. This causes the transmission line to be congested.

By introducing this artificial congestion Firm 2 is able to limit the amount of inexpensive

Table 4.4: Results for Two-Node Network Single Firm BLBP.

Firm 2 Bids and Offers

Bid Demand 2	$(p_{D_2}, \bar{q}_{D_2}) = (100\ \$/\text{MWh}, 5000\ \text{MWh})$
Offer Supplier 2	$(\mathbf{p}_{S_2}, \bar{\mathbf{q}}_{S_2}) = \left(\begin{bmatrix} 100 \\ 100 \end{bmatrix} \$/\text{MWh}, \begin{bmatrix} 3000 \\ 1000 \end{bmatrix} \text{MWh} \right)$
Maximum Transmission Quantity	$\bar{t} = 1000$ MWh

Secondary Variables

Quantity Demand 1	$q_{D_1} = 5000$ MWh
Quantity Demand 2	$q_{D_2} = 5000$ MWh
Quantity Supply 1	$\mathbf{q}_{S_1}^T = \begin{bmatrix} 3000 & 3000 & 0 \end{bmatrix}$ MWh
Quantity Supply 2	$\mathbf{q}_{S_2}^T = \begin{bmatrix} 3000 & 1000 \end{bmatrix}$ MWh
Transmitted Quantity	$t = 1000$ MWh
Transmission Price	$\pi_t = 75$ \$/MWh
MCP Node 1	$\pi_1 = 25$ \$/MWh
MCP Node 2	$\pi_2 = 100$ \$/MWh
Shadow Price Demand 1	$\xi_{D_1} = 0$ \$/MWh
Shadow Price Demand 2	$\xi_{D_2} = 0$ \$/MWh
Shadow Price Supply 1	$\mathbf{z}_{S_1}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ \$/MWh
Shadow Price Supply 2	$\mathbf{z}_{S_2}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$ \$/MWh

Profit

Profit Firm 2	364, 000 $
Profit Demand 1	375,000 $
Profit Supply 1	0 $
Profit Demand 2	0 $
Profit Supplier 2	289,000 $
Profit Transmission Owner	75,000

energy transferred from Node 1 to Node 2. Limiting the transmitted quantity allows the price at Node 1, π_1, to differ from the price at Node 2, π_2. This has two effects: first, it allows Supplier 2 to increase the price of energy at Node 2, and thus increase the profit of Supplier 2, second, it allows the transmission line owner to reap congestion fees. The congestion fees go into effect when t hits its upper bound, $\bar{t}$, causing the dual variable π_t to be nonzero. Since $\pi_t = \pi_2 - \pi_1$, the transmission line owner will have greater profit when π_2 is larger. This also explains why Firm 2 has chosen to place such a high bid of 100 \$/MWh for Demand 2, thus allowing a larger price at Node 2, but giving up all the profit of Demand 2.

Because of this artificial congestion, Firm 2 is able to make a profit on the quantity of inexpensive energy being transmitted from Supplier 1. At the same time, by separating the prices of the two nodes, Firm 2 is not allowing Supplier 1 to raise the price of electricity at Node 1—the price Supplier 1 receives for generating electricity—even in the presence of increased demand from Node 2.

5 A Set of BLBPs arising in the Two-Node Network

There are a total of five market entities in the Two-Node Network: Demand 1, Supply 1, Transmission, Demand 2, and Supply 2. These entities are depicted in Figure 5.1.

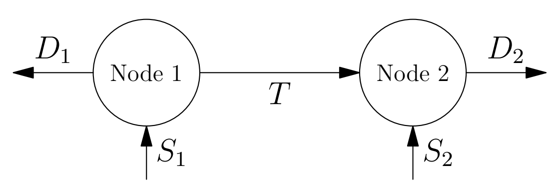

Figure 5.1: A diagram of the two-node network. The five market entities D_1, S_1, T, S_2 and D_2 are labeled.

We have solved the Single Firm Problem when the firm is a vertical utility located at Node 2, controlling Demand 2, Supply 2, and Transmission. By considering firms composed of other permutations of the five market entities it is possible to generate different instances of the Single Firm Problem. There are 31 different instances of possible single-firm ownership of the five entities; from these 31 we have selected eight problems of interest from several representative problem categories. These eight problems are shown in Table 5.1. For example in problem 5 we consider a firm located at Node 1 which controls both supply (Supplier 1) and demand (Demand 1). Note problem 8, that of a vertically integrated firm located at Node 2, is the problem solved in the previous section.

Table 5.1: A set of 8 problems. Entities under the firm's control in the problem are boxed.

Problem Category	#	Market Entities				
Demand Import	1	D_1	$\boxed{D_2}$	$\boxed{T}$	S_1	S_2
Demand Export	2	$\boxed{D_1}$	D_2	$\boxed{T}$	S_1	S_2
Generation Import	3	D_1	D_2	$\boxed{T}$	S_1	$\boxed{S_2}$
Generation Export	4	D_1	D_2	$\boxed{T}$	$\boxed{S_1}$	S_2
Demand & Generation	5	$\boxed{D_1}$	D_2	T	$\boxed{S_1}$	S_2
	6	D_1	$\boxed{D_2}$	T	S_1	$\boxed{S_2}$
Vertical Utility Export	7	$\boxed{D_1}$	D_2	$\boxed{T}$	$\boxed{S_1}$	S_2
Vertical Utility Import	8	D_1	$\boxed{D_2}$	$\boxed{T}$	S_1	$\boxed{S_2}$

6 Numerical Experiments on a set of BLBPs

To analyze the difficulty of solving BLBPs we perform a set of numerical experiments. We take the eight problems described in the previous section and attempt to solve each of them with the nonlinear solvers Snopt [GMS97] and Ipopt [WB06].

These solvers are, of course, designed only to find a solution that is locally optimal. In general, when maximizing a firm's profit, we seek the global maximum. The point of these numerical experiments is to obtain an estimate of the number of local optimum, and how much these local optimum differ from the global optimum on these example BLBPs.

If there are multiple local optimum then the solution to which a solver converges depends on the starting point supplied. Thus, we supply three different starting points to the solvers: $\mathbf{0}$, the point with all variables set to zero, $\mathbf{e}$, the point with all variables set to one, and the point with all variables set to their values at the marginal cost solution. That is, we choose the final starting point by computing the solution to the primal and dual linear programs with all entities in the market bidding marginal cost. The values for the variables at this point are given in Table 4.2.

Tables 6.1, 6.2, and 6.3 show the status of the solver and several quantities of interest in the solutions of Snopt and Ipopt computed from the $\mathbf{0}$, $\mathbf{e}$, and marginal cost starting points. A solver reports a status of: Solved if the final point satisfies the first order optimality conditions, Failed if the current point could not be improved and does not satisfy the optimality conditions, and Infeasible if the solver is not able to find a feasible point after a fixed number of iterations. The default options were used for both Snopt and Ipopt. We do not report iterations or computational time as these were both quite small; the entire set of BLBPs was solved by both Snopt and Ipopt in under a second.

There are several conclusions to be drawn from this data. Perhaps the first thing to note

Table 6.1: Solutions of SNOPT and IPOPT computed from the **0** starting point.

Problem	Solver	Status	t (MWh)	$\bar{t}$ (MWh)	π_1 (\$/MWh)	π_2 (\$/MWh)	Profit (\$)
1	SNOPT	Solved	0	0	25	0	-1.65681e-10
1	IPOPT	Solved	2000	2499.95	35	35	325000
2	SNOPT	Solved	1000	1000	25	36	386000
2	IPOPT	Solved	1000	1000	25	36	386000
3	SNOPT	Solved	1000	1000	25	100	364000
3	IPOPT	Solved	1000	1000	25	100	364000
4	SNOPT	Solved	0	0	100	36	375000
4	IPOPT	Solved	0	0.00991981	100	36	375001
5	SNOPT	Solved	2000	2500	36	36	387000
5	IPOPT	Solved	2000	2500	36	36	387000
6	SNOPT	Solved	0	2500	25	0	0
6	IPOPT	Solved	2000	2500	34.9881	34.9881	355024
7	SNOPT	Solved	1000	1000	25	36	386000
7	IPOPT	Solved	2000	2499.62	36	36	387000
8	SNOPT	Solved	0	0	25	0	0
8	IPOPT	Solved	1000	1000	25	84.3839	364000

Table 6.2: Solutions of SNOPT and IPOPT computed from the **e** starting point.

Problem	Solver	Status	t (MWh)	$\bar{t}$ (MWh)	π_1 (\$/MWh)	π_2 (\$/MWh)	Profit (\$)
1	SNOPT	Solved	1000	1000	25	25	300000
1	IPOPT	Solved	1000	1000	25	36	331000
2	SNOPT	Solved	2000	2000	25	36	22000
2	IPOPT	Solved	1000	1000	25	36	386000
3	SNOPT	Solved	1000	1000	25	100	300000
3	IPOPT	Solved	1000	1000	25	100	364000
4	SNOPT	Solved	0	0	100	36	375000
4	IPOPT	Solved	0	0.0274558	100	36	375000
5	SNOPT	Solved	1000	2500	36	36	386000
5	IPOPT	Solved	2000	2500	36	36	387000
6	SNOPT	Failure	—	—	—	—	—
6	IPOPT	Solved	2000	2500	34.9952	34.9952	355010
7	SNOPT	Solved	2000	2000	36	36	387000
7	IPOPT	Solved	2000	2000.68	36	36	387000
8	SNOPT	Failure	—	—	—	—	—
8	IPOPT	Solved	1000	1000	25	99.6845	364000

Table 6.3: Solutions of SNOPT and IPOPT computed from the marginal cost solution.

Problem	Solver	Status	t (MWh)	$\bar{t}$ (MWh)	π_1 (\$/MWh)	π_2 (\$/MWh)	Profit (\$)
1	SNOPT	Solved	2000	2500	35	35	325000
1	IPOPT	Solved	2000	2498.5	35	35	325000
2	SNOPT	Solved	2000	2500	35	35	325000
2	IPOPT	Solved	2000	2000	35	36	327000
3	SNOPT	Solved	2000	2500	35	35	30000
3	IPOPT	Solved	2000	2000	35	100	355000
4	SNOPT	Solved	2000	2500	36	36	67000
4	IPOPT	Solved	2000	2492.41	36	36	67000
5	SNOPT	Solved	2000	2500	36	36	387000
5	IPOPT	Solved	2000	2500	36	36	387000
6	SNOPT	Solved	2000	2500	35	35	355000
6	IPOPT	Solved	2000	2500	32.7008	32.7008	359598
7	SNOPT	Infeasible	—	—	—	—	—
7	IPOPT	Solved	2000	2000.72	36	36	387000
8	SNOPT	Solved	2000	2500	35	35	355000
8	IPOPT	Solved	2000	2002.19	34.9999	34.9999	355000

is that for every problem, with the exceptions of problems 5 and 7, both solvers compute different values for the firm's profit when given different starting points. Problems 5 and 7 are exceptions because in these problems the firm is located completely at Node 1 and thus the effects of the transmission line do not come into play. Thus, we see that indeed these BLBPs do contain multiple local optimum.

When starting from the **0** point SNOPT computes solutions with a market clearing price of zero at Node 2 for problems 1, 6 and 8. As a result no electricity is transmitted to Node 2, and the bids of Demand 2 and offers of Supplier 2 are not accepted. Therefore, the firms in these problems earn no profit. Note, however, that it is possible for these firms to earn quite a large profit.

These zero profit solutions disappear when starting from the point **e**. However, from this starting point SNOPT fails to compute a solution for problems 6 and 8.

From this data we see that for problems 1, 2, 3 and 8, it is advantageous for the firm controlling the transmission line to force the line to become artificially congested. When started from points **0** or **e** both SNOPT and IPOPT converge to solutions with artificial congestion. However, when started from the marginal cost solution, both solvers converge to solutions with no artificial congestion. As a result, profits computed when starting from the marginal cost solution tend to be lower. In addition, when starting from the marginal cost solution, SNOPT is unable to find a feasible point for problem 7.

6.1 A Homotopy Method

In an effort to obtain the global optimum, and avoid the problems associated with a single starting point we employ a homotopy method. In the homotopy method we solve a sequence of problems in the form

$$\begin{aligned}\mathrm{BLBP}(\mu) = \text{maximize } & f(\mathbf{p}, \mathbf{q}, \boldsymbol{\pi}) \\ \text{subject to } & A\mathbf{q} = \mathbf{b}, \quad A^T\boldsymbol{\pi} + \mathbf{s} = \mathbf{p} \\ & \mathbf{q}^T\mathbf{s} = \mu \\ & \mathbf{0} \leq \mathbf{q}, \mathbf{s}.\end{aligned} \tag{6.1}$$

Here $\mu > 0$ is a parameter that controls the duality gap of, or the precision to which we solve, the lower-level linear program. Let $\mathbf{p}^\star(\mu), \mathbf{q}^\star(\mu)$ denote a locally optimal solution to BLBP(μ). In the homotopy method, we first fix a value of $\mu_k > 0$, solve BLBP(μ_k) to some loose precision, and then set $\mu_{k+1} = \rho\mu_k$, for $\rho < 1$. The values of $\mathbf{p}^\star(\mu_k)$ and $\mathbf{q}^\star(\mu_k)$ are used as the starting point for the the new problem BLBP(μ_{k+1}). After n steps, when $\mu_n < \epsilon$, the solution $\mathbf{p}^\star(\mu_n), \mathbf{q}^\star(\mu_n)$ satisfies the nonlinear constraints of the original problem to within a tolerance ϵ.

When solving this sequence of problems we hope to trace out a continuous curve to the solution of the original problem. Of course, because these problems are nonconvex we cannot be assured such a curve is continuous or even exists. Table 6.4 shows the results of applying the homotopy method with the solvers SNOPT and IPOPT. We use an initial value of $\mu_0 = 1$, set $\rho = 1/2$ and solve $n = 21$ problems to achieve a tolerance of $\epsilon = \texttt{1e}{-}\texttt{6}$.

The results of the homotopy method are encouraging. With IPOPT the profits obtained via the homotopy method are as good or better than those obtained from all other starting points on all problems (except for problem 1). Furthermore, when using the homotopy method

Table 6.4: Solutions of Snopt and Ipopt computed with the Homotopy Method

Problem	Solver	Status	t (MWh)	$\bar{t}$ (MWh)	π_1 (\$/MWh)	π_2 (\$/MWh)	Profit (\$)
1	Snopt	Solved	1000	1000.05	25	25	300000
1	Ipopt	Solved	2000	2498.5	35	35	325000
2	Snopt	Solved	1000	1000	25	36	386000
2	Ipopt	Solved	1000	1000	25	36	386000
3	Snopt	Solved	1000	1000	25	100	364000
3	Ipopt	Solved	1000	1000	25	100	364000
4	Snopt	Solved	0	0	100	36	375000
4	Ipopt	Solved	2.7721e-09	1264.14	100	36	375002
5	Snopt	Solved	1000	2500	36	36	386000
5	Ipopt	Solved	2000	2500	36	36	387000
6	Snopt	Solved	1000	2500	25	25	300000
6	Ipopt	Solved	2000	2500	35	35	355000
7	Snopt	Solved	1000	1000	25	36	386000
7	Ipopt	Solved	2000	2000.04	36	36	387000
8	Snopt	Solved	1000	1000	25	100	300000
8	Ipopt	Solved	1000	1000	25	71.619	364000

both solvers correctly identify solutions that contain artificial congestion. In addition, with the homotopy method, SNOPT does not get stuck at zero profit local maximums, fail, or incorrectly classify problem 7 as infeasible.

6.2 Alternative Optimum

Another crucial aspect of these BLBPs, revealed by the data in Tables 6.1, 6.2, 6.3, and 6.4, is the presence of alternative optimal solutions. Here, we define alternative optimal solutions to be two or more solutions with the same objective value but different values for the primary decision variables (*i.e.*, the nodal prices, or the bids and offers that the firm controls).

The most prominent example of alternative optimum occurs in problem 8. In Section 4, starting points $\mathbf{0}$, $\mathbf{e}$, and the homotopy method we calculate a profit of 364000 dollars. Each of these four calculations yield different values for the market clearing price at Node 2, π_2: 100 \$/MWh, 84.4 \$/MWh, 99.6 \$/MWh, and 71.6 \$/MWh.

This can be explained by examining the objective function for problem 8,

$$\underbrace{(\pi_2 \mathbf{e}^T - \mathbf{c}_{S_2}^T)\mathbf{q}_{S_2}}_{\text{Supplier 2}} + \underbrace{(\pi_2 - \pi_1)t}_{\text{Transmission}} + \underbrace{(r_{D_2} - \pi_2)q_{D_2}}_{\text{Demand 2}}.$$

Note that Firm 2 has three different sources of profit; profit made from Supplier 2, the transmission line, and from Demand 2. Each of these three terms depend on the market clearing price, π_2. By choosing its bids and offers to inflate π_2 Firm 2 earns profit from Supplier 2 and Transmission, at a loss of profit from Demand 2. By choosing its bids and offers to deflate π_2 Firm 2 earns profit from Demand 2, at a loss of profit from Supplier 2 and Transmission. These three terms balance each other, and thus there is a whole set of offer and bid combinations for Firm 2, all of which yield a profit of 364000 dollars.

7 Conclusions

In this work we have formulated and solved several BLBPs arising from electric power markets. These BLBPs all have a very similar structure. As there name suggests, they contain a bilinear, and hence nonlinear and nonconvex objective function, and a single bilinear constraint arising from the optimality conditions of the lower-level linear program. In the past BLBPs have been solved by approximating these bilinear terms in the objective and constraints, and using special ordered sets to transform the BLBP into a linear program. This work shows that there are no inherit numerical difficulties or other issues in retaining the nonlinear objective and constraints and solving these BLBPs with a fully nonlinear solver like Snopt or Ipopt.

Unfortunately, these nonlinear solvers will not always compute the global maximum (nor should we expect them to). For a reasonable solution with an accurate estimate of the profit to be obtained multiple starting points must be used, or these nonlinear solvers need to be coupled with global optimization techniques such as homotopy methods or branch and bound algorithms.

This worked used nonlinear solvers to solve the nonlinear formulation of a BLBP. Future work should examine solvers, such as Miles and Path, which explicitly deal with complementarity constraints, and investigate their performance on these BLBPs. Complementarity pivoting algorithms should also be investigated.

People solve BLBPs every day armed with merely pencil and paper; usually they do this by checking a small number of points and reasoning about the maximum profit that can be obtained. Future work on algorithms for solving BLBPs needs to be guided by this intuition and reasoning.

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